Journal of Applied Mechanics and Technical Physics, Vol. 42, No. 6, pp. 1038–1042, 2001

PLANE ELASTIC PROBLEM FOR AN INHOMOGENEOUS LAYERED BODY

A. E. Alekseev, V. V. Alekhin, and B. D. Annin

UDC 539.3

A plane elastic problem for an inhomogeneous elastic layered body bounded by equidistant convex curves is considered. A numerical algorithm for solving the problem is proposed and implemented.

Introduction. Various procedures of constructing equations governing elastic deformation of multilayered structures are considered in [1–3].

In the present paper, to construct equations that describe elastic deformation of a layered body, we use the results of [4–7], which allows us to formulate correct conjugation conditions for stresses and displacements at the interlayer boundaries. The results obtained can be used to optimize layered structures [8].

1. Curvilinear Coordinate System. Let L be a sufficiently smooth closed convex curve bounding a domain D. We assume that the radius of curvature at each point of L is not smaller than ρ_*.

Let R be an arbitrary point of the curve L (Fig. 1), $\boldsymbol{t}$ and $\boldsymbol{n}$ be, respectively, the unit tangent and normal vectors to L at the point R, and β be the angle between the tangent $\boldsymbol{t}$ and the x axis such that $\beta = \pi/2$ at the point of intersection of L and the x axis. Let the origin (point O) lie inside the domain D. The equations of the oval L can be written in the parametric form [9]:

$$x_L(\beta) = \frac{dF(\beta)}{d\beta}\cos\beta + F(\beta)\sin\beta, \qquad y_L(\beta) = \frac{dF(\beta)}{d\beta}\sin\beta - F(\beta)\cos\beta, \qquad \frac{\pi}{2} \leqslant \beta \leqslant \frac{5\pi}{2}.$$

Here $x_L(\beta)$ and $y_L(\beta)$ are the Cartesian coordinates of the point R and $F(\beta) \geqslant 0$ is the periodic (with a period of 2π) support function of the contour L (distance from the point O to the tangent $\boldsymbol{t}$). Each value of $\beta \in [\pi/2, 5\pi/2)$ corresponds to one and only one point $R \in L$. The radius of curvature of L is $\rho = \rho(\beta) = F(\beta) + d^2F(\beta)/d\beta^2 \geqslant \rho_*$, and the unit vectors $\boldsymbol{t}$ and $\boldsymbol{n}$ have the form $\boldsymbol{t} = (\cos\beta, \sin\beta)$ and $\boldsymbol{n} = (\sin\beta, -\cos\beta)$.

We consider the orthogonal curvilinear coordinate system (α, β) induced by the contour L with the support function $F(\beta)$:

$$\begin{aligned} x = x(\alpha,\beta) &= \frac{dF(\beta)}{d\beta}\cos\beta + (F(\beta)+\alpha)\sin\beta, \\ y = y(\alpha,\beta) &= \frac{dF(\beta)}{d\beta}\sin\beta - (F(\beta)+\alpha)\cos\beta, \end{aligned} \qquad \frac{\pi}{2} \leqslant \beta \leqslant \frac{5\pi}{2}, \quad \alpha \geqslant 0. \tag{1.1}$$

The Jacobian of coordinate transformation has the form $J(\alpha, \beta) = D(x, y)/D(\alpha, \beta) = \rho(\beta) + \alpha > 0$. It follows from (1.1) that the coordinate lines $\alpha = \text{const}$ are equidistant curves with the support function $F_\alpha = F_\alpha(\beta) = F(\beta) + \alpha$, and the coordinate lines $\beta = \text{const}$ form a family of straight lines normal to L.

2. Equations of Plane Elastic Problem in the Curvilinear Coordinate System (α, β). We write the equations of the plane elastic problem in the orthogonal curvilinear coordinate system (α, β).

The stresses $\sigma_{\alpha\alpha}$, $\sigma_{\alpha\beta}$, and $\sigma_{\beta\beta}$ satisfy the equations of equilibrium

$$\frac{\partial\sigma_{\alpha\alpha}}{\partial\alpha} + \frac{1}{\rho+\alpha}\frac{\partial\sigma_{\alpha\beta}}{\partial\beta} + \frac{\sigma_{\alpha\alpha} - \sigma_{\beta\beta}}{\rho+\alpha} = 0, \qquad \frac{\partial\sigma_{\alpha\beta}}{\partial\alpha} + \frac{1}{\rho+\alpha}\frac{\partial\sigma_{\beta\beta}}{\partial\beta} + \frac{2\sigma_{\alpha\beta}}{\rho+\alpha} = 0.$$

The strain tensor is expressed in terms of the displacement-vector components $\boldsymbol{u} = u_\alpha\boldsymbol{n} + u_\beta\boldsymbol{t}$ (u_α and u_β are functions of α and β, respectively):

Lavrent'ev Institute of Hydrodynamics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090.
Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 42, No. 6, pp. 136–141, November–December, 2001. Original article submitted June 25, 2001.

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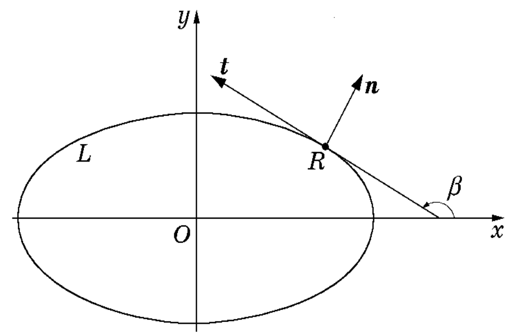

Fig. 1

$$e_{\alpha\alpha} = \frac{\partial u_\alpha}{\partial \alpha}, \quad e_{\beta\beta} = \frac{1}{\rho+\alpha}\,\frac{\partial u_\beta}{\partial \beta} + \frac{1}{\rho+\alpha}\,u_\alpha, \quad e_{\alpha\beta} = \frac{1}{2}\Big(\frac{\partial u_\beta}{\partial \alpha} + \frac{1}{\rho+\alpha}\,\frac{\partial u_\alpha}{\partial \beta} - \frac{u_\beta}{\rho+\alpha}\Big).$$

The strains are related to the stresses by Hooke's law:

$$e_{\alpha\alpha} = \frac{1-\nu^2}{E}\Big(\sigma_{\alpha\alpha} - \frac{\nu}{1-\nu}\,\sigma_{\beta\beta}\Big), \quad e_{\beta\beta} = \frac{1-\nu^2}{E}\Big(\sigma_{\beta\beta} - \frac{\nu}{1-\nu}\,\sigma_{\alpha\alpha}\Big),$$
$$2e_{\alpha\beta} = \sigma_{\alpha\beta}/\mu, \quad 2\mu = E/(1+\nu). \tag{2.1}$$

Here E and μ are Young's modulus and shear modulus, respectively, and ν is Poisson's ratio.

3. Equations of Elastic Deformation of a Cylindrical Shell of Oval Cross Section. We consider an infinitely long body of thickness $2h$ bounded by the coordinate surfaces α_1, α_2, β_1, and β_2: $0 < \alpha_1 < \alpha_2 = \alpha_1 + 2h$ and $\pi/2 \leqslant \beta_1 < \beta_2 < 5\pi/2$. We introduce the coordinate $\xi \in [-1,1]$ related to α by the formula $\alpha = (\alpha_1+\alpha_2)/2 + \xi(\alpha_2-\alpha_1)/2$.

Following [4–7], we approximate the stresses by truncated series in Legendre polynomials $P_k(\xi)$:

$$2h\sigma_{\beta\beta} = N + (3M/h)P_1(\xi), \quad \sigma_{\alpha\alpha} = p_0 + \Delta p P_1(\xi),$$
$$2h\sigma_{\alpha\beta} = Q + 2h\Delta q P_1(\xi) + (2hq_0 - Q)P_2(\xi), \tag{3.1}$$
$$\Delta q = (q^+ - q^-)/2, \quad q_0 = (q^+ + q^-)/2, \quad \Delta p = (p^+ - p^-)/2, \quad p_0 = (p^+ + p^-)/2.$$

Here $N = h\int\limits_{-1}^{1} \sigma_{\beta\beta}\,d\xi$ is the force, $M = h^2\int\limits_{-1}^{1} \sigma_{\beta\beta}\xi\,d\xi$ is the moment, $Q = h\int\limits_{-1}^{1} \sigma_{\alpha\beta}\,d\xi$ is the transverse shear force, and $q^\pm = \sigma_{\alpha\beta}|_{\xi=\pm1}$ and $p^\pm = \sigma_{\alpha\alpha}|_{\xi=\pm1}$ are, respectively, the shear and normal stresses at the boundary surfaces $\alpha = \alpha_1$, $\alpha = \alpha_2$.

The displacements u_β and u_α are approximated by the truncated series

$$u_\beta = u + \psi P_1(\xi) + (u_0 - u)P_2(\xi) + (\Delta u - \psi)P_3(\xi), \qquad u_\alpha = v + \Delta v P_1(\xi) + (v_0 - v)P_2(\xi),$$
$$\Delta u = (u^+ - u^-)/2, \quad u_0 = (u^+ + u^-)/2, \quad \Delta v = (v^+ - v^-)/2, \quad v_0 = (v^+ + v^-)/2.$$

Here $u = \frac{1}{2}\int\limits_{-1}^{1} u_\beta\,d\xi$ is the β-displacement averaged through the thickness, $v = \frac{1}{2}\int\limits_{-1}^{1} u_\alpha\,d\xi$ is the α-displacement averaged through the thickness, and $u^\pm = u_\beta|_{\xi=\pm1}$ and $v^\pm = u_\alpha|_{\xi=\pm1}$ are, respectively, the tangential and normal displacements at the boundary surfaces $\alpha = \alpha_1$, $\alpha = \alpha_2$.

The strains are approximated by the truncated series

$$e_{\beta\beta} = \frac{1}{\rho_0}\Big(\frac{du}{d\beta} + v + \frac{d\psi}{d\beta}\,P_1(\xi)\Big), \qquad e_{\alpha\alpha} = \frac{\Delta v}{h} + 3\,\frac{v_0 - v}{h}\,P_1,$$
$$2e_{\alpha\beta} = \frac{\Delta u}{h} + \frac{1}{\rho_0}\Big(\frac{dv}{d\beta} - u\Big) + 3P_1\,\frac{u_0 - u}{h} + 5P_2\,\frac{\Delta u - \psi}{h}. \tag{3.2}$$

Here $\rho_0 = \rho + h = (\alpha_1 + \alpha_2)/2$.

Substituting stresses (3.1) and strains (3.2) into Hooke's law (2.1) and equating the coefficients of the same Legendre polynomials $P_k(\xi)$, we obtain

$$\frac{1}{\rho_0}\Big(\frac{du}{d\beta}+v\Big)=\frac{N}{2hE^*}-\frac{\nu^* p_0}{E^*},\quad \frac{1}{\rho_0}\,\frac{d\psi}{d\beta}=\frac{3M}{2h^2E^*}-\frac{\nu^*\Delta p}{E^*},$$
$$\frac{1}{\rho_0}\Big(\frac{dv}{d\beta}-u\Big)+\frac{\Delta u}{h}=\frac{1}{2h\mu}\,Q; \tag{3.3}$$

$$3\,\frac{u_0-u}{h}=\frac{\Delta q}{\mu},\qquad 5\,\frac{\Delta u-\psi}{h}=\frac{q_0}{\mu}-\frac{Q}{2h\mu},$$
$$\frac{\Delta v}{h}=\frac{p_0}{E^*}-\frac{\nu^*}{E^*}\,\frac{N}{2h},\qquad 3\,\frac{v_0-v}{h}=\frac{\Delta p}{E^*}-\frac{\nu^*}{E^*}\,\frac{3}{2h^2}M. \tag{3.4}$$

Here $E^*=E/(1-\nu^2)$ and $\nu^*=\nu/(1-\nu)$.

The equations of equilibrium have the form

$$\frac{1}{\rho_0}\Big(\frac{dN}{d\beta}+Q\Big)+2\Delta q=0,\quad \frac{1}{\rho_0}\Big(\frac{dQ}{d\beta}-N\Big)+2\Delta p=0,\quad \frac{1}{\rho_0}\,\frac{dM}{d\beta}-Q+2hq_0=0. \tag{3.5}$$

For given external stresses $\{p^\pm,q^\pm\}$, Eqs. (3.3) and (3.5) constitute a closed system of ordinary differential equations for unknown functions N, M, Q, u, ψ, and v. The functions $u^\pm$ and $v^\pm$ (displacements at the boundaries $\xi=\pm1$) are determined from the algebraic equations (3.4).

4. Equations of a Layered Body Composed of Equidistant Layers. We consider a curve L_0 with a support function $F_0(\beta)$. The curves L_i ($i=\overline{1,n}$) with the support functions $F_i(\beta)=F_{i-1}(\beta)+2h^i$ form a family of equidistant curves with the distance $2h^i$ between the neighboring curves L_i and L_{i-1}. The radius of curvature of L_i is $\rho_i=\rho_{i-1}+2h^i$ ($i=\overline{1,n}$).

Let B be a continuous body composed of individual layers B_i ($i=\overline{1,n}$) bounded by the curves L_{i-1} and L_i. The contour L_i is the interface between the layers B_i and B_{i+1} ($i=\overline{1,n}$). The line L_0 coincides with L.

We denote the quantities corresponding to the layer B_i by the superscript i. Using the algebraic equations (3.4), we obtain the expressions for $(p^+)^i$, $(q^+)^i$, $(v^+)^i$, and $(u^+)^i$:

$$(p^+)^i=-\frac{3(E^*)^i}{h^i}\,(v^-)^i-2(p^-)^i+\frac{3(E^*)^i}{h^i}\,v^i+\frac{3\nu^i}{2h^i}\Big(N^i-\frac{M^i}{h^i}\Big),$$
$$(q^+)^i=\frac{15\mu^i}{h^i}\,(u^-)^i+4(q^-)^i+\frac{15\mu^i}{h^i}\,(-u^i+\psi^i)-\frac{3Q^i}{2h^i},$$
$$(v^+)^i=-2(v^-)^i+3v^i-\frac{h^i}{(E^*)^i}\,(p^-)^i+\frac{(\nu^*)^i}{2(E^*)^i}\Big(N^i-\frac{3M^i}{h^i}\Big),$$
$$(u^+)^i=4(u^-)^i+\frac{h^i}{\mu^i}\,(q^-)^i-3u^i+5\psi^i-\frac{Q^i}{2\mu^i}. \tag{4.1}$$

The following conditions should be satisfied at the interfaces L_i ($i=\overline{1,n-1}$):

$$(p^+)^i=(p^-)^{i+1},\quad (q^+)^i=(q^-)^{i+1},\quad (v^+)^i=(v^-)^{i+1},\quad (u^+)^i=(u^-)^{i+1}. \tag{4.2}$$

We confine ourselves to the case where the stresses

$$(q^-)^1=Q_0,\quad (q^+)^n=Q_n,\quad (p^-)^1=P_0,\quad (p^+)^n=P_n \tag{4.3}$$

are specified at the face lines L_0 and L_n of the layered body B. From the system of linear algebraic equations (4.1)–(4.3), we obtain

$$(p^+)^i=A_1^iP_n+A_2^iP_0+\sum_{k=1}^{i}(a_{1k}^iv^k+a_{2k}^iN^k+a_{3k}^iM^k),$$

$$(v^+)^i=B_1^iP_n+B_2^iP_0+\sum_{k=1}^{i}(b_{1k}^iv^k+b_{2k}^iN^k+b_{3k}^iM^k),$$

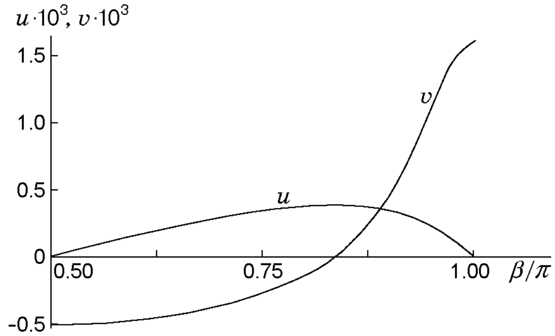

Fig. 2

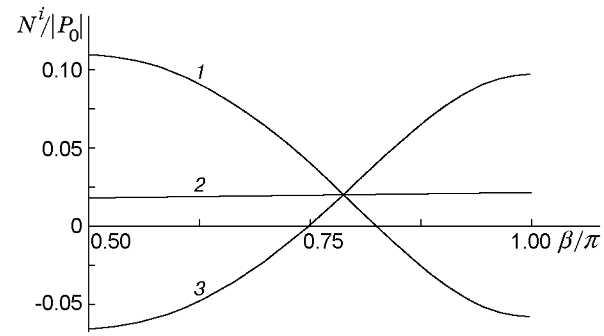

Fig. 3

$$
\begin{aligned}
(q^+)^i &= C_1^i Q_n + C_2^i Q_0 + \sum_{k=1}^{i} (c_{1k}^i u^k + c_{2k}^i \psi^k + c_{3k}^i Q^k), \\
(u^+)^i &= D_1^i Q_n + D_2^i Q_0 + \sum_{k=1}^{i} (d_{1k}^i u^k + d_{2k}^i \psi^k + d_{3k}^i Q^k), \\
V_n = (v^+)^n &= B_1^n P_n + B_2^n P_0 + \sum_{k=1}^{n} b_{1k}^n v^k + b_{2k}^n N^k + b_{3k}^n M^k), \\
V_0 = (v^-)^1 &= B_1^0 P_n + B_2^0 P_0 + \sum_{k=1}^{n} (b_{1k}^0 v^k + b_{2k}^0 N^k + b_{3k}^0 M^k), \\
U_n = (u^+)^n &= D_1^n Q_n + D_2^n Q_0 + \sum_{k=1}^{n} d_{1k}^n u^k + d_{2k}^n \psi^k + A_{3k}^n Q^k), \\
U_0 = (u^-)^1 &= D_1^0 Q_n + D_2^0 Q_0 + \sum_{k=1}^{n} (d_{1k}^0 u^k + d_{2k}^0 \psi^k + d_{3k}^0 Q^k).
\end{aligned}
\tag{4.4}
$$

Substitution of Eqs. (4.4) into Eqs. (3.3) and (3.5) yields the system of linear ordinary differential equations

$$\frac{d\boldsymbol{X}}{d\beta} = A\boldsymbol{X} + B, \tag{4.5}$$

where $\boldsymbol{X} = (u^1, \ldots, u^n, \psi^1, \ldots, \psi^n, Q^1, \ldots, Q^n, v^1, \ldots, v^n, N^1, \ldots, N^n, M^1, \ldots, M^n)$ is the vector of unknown functions.

5. Examples of Solutions. As an example, we consider elastic deformation of an infinite layered tube with an elliptic internal contour. In this case, the support function $F(\beta)$ and the radius of curvature $\rho(\beta)$ have the form

$$F(\beta) = \sqrt{a^2 \sin^2 \beta + b^2 \cos^2 \beta}, \quad \rho(\beta) = a^2 b^2 / F(\beta), \tag{5.1}$$

where a and b are the semiaxes of the ellipse.

The tube is subjected to internal pressure. The boundary conditions (4.3) become

$$Q_0 = 0, \quad Q_n = 0, \quad P_0 = -100 \text{ MPa}, \quad P_n = 0. \tag{5.2}$$

It follows from (5.1) and (5.2) that the problem is symmetric about the coordinate axes Ox and Oy. Therefore, the differential equations (4.5) are supplemented by the boundary conditions

$$X_i(\pi/2) = 0, \qquad X_i(\pi) = 0, \qquad i = \overline{1, 3n}. \tag{5.3}$$

The boundary-value problem (4.5), (5.3) is solved numerically by Godunov's method [10].

To verify the algorithm proposed, we considered the deformation of a single-layered tube ($n = 1$). In this case, Eqs. (3.3)–(3.5) yield the exact solution for the hoop force N and transverse shear force Q:

$$N = -P_0 F(\beta), \qquad Q = P_0 \frac{dF(\beta)}{d\beta}. \tag{5.4}$$

The calculations were performed for the following physical and geometrical parameters: $E = 2.1 \cdot 10^5$ MPa, $\nu = 0.3$ (steel), $h = 0.05$ m, $a = 1.5$ m, and $b = 0.5$ m.

The maximum differences between the values of N and Q obtained by the numerical method and those calculated by formulas (5.4) are 0.1 and 0.15%, respectively. Figure 2 shows the normal (v) and tangential (u) displacements of the mid-surface versus the angle β.

We consider the problem of elastic deformation of a three-layered tube with the following parameters: $E = 2.1 \cdot 10^5$ MPa and $\nu = 0.3$ (steel) for the internal layer, $E = 2.7 \cdot 10^3$ MPa and $\nu = 0.27$ (spheroplastic) for the middle layer, and $E = 1.2 \cdot 10^5$ MPa and $\nu = 0.32$ (titanium alloy) for the external layer. The geometrical parameters were $h^i = 1/60$ m ($i = \overline{1,3}$), $a = 1.1$ m, and $b = 0.9$ m. Figure 3 shows the forces N^i versus the angle β for $i = 1$, 2, and 3 (curves 1–3, respectively). The calculations were performed on a Pentium-II computer.

This work was partly supported by the Ministry of Education of Russian Federation (Grant No. E-00-4.0-120).

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